

Structure of Six–Dimensional Microstate Geometries

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ABSTRACT: We investigate the structure of smooth and horizonless microstate geometries in six dimensions, in the spirit of the five-dimensional analysis of Gibbons and Warner [[arXiv:1305.0957](https://arxiv.org/abs/1305.0957)]. In six dimensions, which is the natural setting for horizonless geometries with the charges of the D1-D5-P black hole, the natural black objects are strings and there are no Chern-Simons terms for the tensor gauge fields. However, we still find that the same reasoning applies: in absence of horizons, there can be no smooth stationary solutions without non-trivial topology. We use topological arguments to describe the Smarr formula in various examples: the uplift of the five-dimensional minimal supergravity microstates to six dimensions, the two-charge D1-D5 microstates, and the non-extremal JMaRT solution. We also discuss D1-D5-P superstrata and confirm that the Smarr formula gives the same result as for the D1-D5 supertubes which are topologically equivalent.

KEYWORDS: Black Holes in String Theory, Black Holes

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1 Introduction

The fuzzball programme argues that extended objects of string theory alter the horizon of black holes quite drastically.¹ Classical solutions to the supergravity equations of motion have played a significant role in shaping fuzzball discussions. These ‘microstate geometries’ are smooth, globally hyperbolic, and solitonic stationary solutions that carry the same charges as a black hole and are argued to correspond to the gravitational interpretation of certain black hole microstates.

¹For reviews, see [1–6]. Related arguments instigated the recent firewall discussion [7–9].

The very existence of such smooth solitonic solutions was a bit puzzling. Based on Smarr formulas, many no-go theorems have been proven in the past for regular, stationary solutions in four dimensions, also in supergravity [10]. They can be summarized through the slogan: *no solitons without horizons*. However, based on explicit construction, it has become clear that there are many supersymmetric horizonless smooth geometries with charges supported by topological fluxes. Also, families of near-supersymmetric microstate geometries are supported by fluxes on non-trivial two-cycles [11, 12].

The seemingly paradoxical existence of these horizonless microstate geometries was further clarified by Gibbons and Warner in [13]. They revisited the Smarr formula in detail for five-dimensional asymptotically flat supergravity solutions and explained the mechanism that supports mass in a stationary, horizonless soliton. Besides receiving contributions from horizons, the Komar mass is also affected by previously neglected terms arising from spatial sections with non-trivial second cohomology. This is possible due to the existence of Chern-Simons terms in the action and cohomologically supported fluxes. The key slogan must be modified to *no solitons without horizons or topology and fluxes*, which then holds regardless of supersymmetry and is in particular also valid for stationary solutions carrying the charges of a black hole with non-zero Hawking temperature. This has been further corroborated by the similar M-theory analysis of [14] and its compactification to five dimensions.

In this paper we want to discuss the topological Smarr formula in the six-dimensional arena appropriate to current microstate geometry research for the three-charge black hole. In addition, we want to explore non-extremal solutions, which have not been discussed before from the viewpoint of considering their topological contributions to the mass.²

The three charge black hole has five non-compact dimensions. It can, for instance, be obtained in IIB string theory on $T^4 \times S^1$ with D1-D5-P charges on the internal directions. The string theory microstates dual to this black are given in the D1-D5 orbifold CFT. The discussion of the Smarr formula of [13] concentrated on the five-dimensional microstate geometries, which fit in five-dimensional $\mathcal{N} = 1$ supergravity with three vector fields in $U(1)^3$, after dimensional reduction on $T^4 \times S^1$. However, the geometric interpretation of the D1-D5-P microstates can in principle excite the full ten-dimensional spacetime and does not have to allow a five-dimensional truncation. Indeed, the microstate geometries for the two-charge D1-D5 black hole, or ‘supertubes’, depend on functions of one variable, the coordinate along the S^1 . These functions describe a profile in the 8 spatial components orthogonal to S^1 (four non-compact dimensions [16–18] and four torus directions [19]) and all IIB supergravity fields are excited.

A similar story is expected to hold for the microstates of the three-charge D1-D5-P black hole. As argued in [20], the generic microstate geometries in the D1-D5-P frame are expected to be described by so-called *superstrata*. These should be obtained from adding momentum modes on two-charge D1-D5 supertubes and depend on functions of two variables. Their solution space might even give a leading contribution to the Bekenstein-Hawking entropy of the black hole [21]. Arbitrary superstrata excitations are most likely computationally beyond our reach and therefore the search for solutions has been focused

²With the exception of the BPS-bound violating solution of [15] discussed in [13].

on keeping the T^4 rigid. The rigorous treatment of [22] shows that any solution sharing the supersymmetries of the D1-D5-P brane system in IIB then fits in six-dimensional $\mathcal{N} = (1, 0)$ supergravity with *two* tensor multiplets, confirming earlier results of [19, 23–25]³. This may be somewhat surprising, since the D1-D5-P black hole and many of its microstate geometries only excite one tensor multiplet. Nonetheless, this extra tensor multiplet helps to smoothen singularities in supergravity solutions [29, 30] and was in fact a key ingredient for the first successful construction of a superstratum [31].

We extend the five-dimensional results of [13] to the framework relevant for the more abundant conjectured six-dimensional solutions. The Komar integral that gives the conserved charge for a Killing vector again has a contribution for non-trivial topology, depending on the third cohomology of space. There are several new features in our discussion:

- *New asymptotics:* The natural black object in our discussion is the six-dimensional D1-D5-P black string, where the string wraps the compact S^1 (compactification along S^1 gives the three-charge black hole). Hence we do not consider asymptotically flat $\mathbb{R}^{5,1}$ spacetimes, but rather focus on $\mathbb{R}^{4,1} \times S^1$ asymptotics. This also implies a different relation between the Komar integral and Smarr formula. For an extended object such as the black string, we cannot just relate the appropriate Komar integrals to the ADM mass, but rather to combinations of both the integrated energy density and tension of the string [32–34].
- *For supersymmetric solutions:* In six dimensions these always have a *null* Killing vector [35] but not necessarily a timelike one as in five dimensions. The Komar integral for a null Killing vector does not give the ADM mass, but rather a ‘null charge’. We discuss the interpretation of this charge and its relation to the mass. As examples, we discuss the uplift of the 5D microstate geometries of [13], D1-D5 two-charge geometries, and D1-D5-P superstrata. We find that the Komar integral for the null charge is independent of the momentum charge P . This is natural as D1-D5-P superstrata and D1-D5 supertubes share the same topology: both describe “wiggles” of a topological three-sphere.
- *For non-supersymmetric solutions:* We explore the JMaRT solutions [36] which have a timelike Killing vector. The Komar integral gives the ADM mass. The JMaRT solitons are only smooth in six dimensions (not in five or four). As in the D1-D5 solutions, the flux through a non-contractible S^3 supports the charges.

The rest of this paper is organized as follows. In section 2 we discuss the Komar integrals and the Smarr formula. We revisit brane-like solutions with compact dimensions and their energy densities and tensions. We normalize the Komar integral in terms of these physical quantities and highlight the difference between using timelike and null Killing vectors. Then we discuss the Komar integral in six-dimensional supergravity. We illustrate the general formula with supersymmetric examples in section 3, and the non-supersymmetric JMaRT solutions [36] in section 4. The supersymmetric examples include

³The earliest three-charge geometries were six-dimensional by construction [26–28], but lacked the intricacy of superstrata that has the potential of a solution space with large entropic contribution.

the uplift of the five-dimensional multi-center solutions discussed in [13] and the D1-D5 Lunin-Mathur geometries [16–18]; we also comment on D1-D5-P superstrata. We end with a discussion in section 5. Appendix A contains the details of the uplift to six dimensions of five-dimensional multi-center solutions and in Appendix B we give the details of the truncation of IIB supergravity to six-dimensional supergravity with two tensor multiplets.

2 Smarr Formula in Six Dimensions

We discuss Komar integrals, the relation to the energy and tension of a solution, a Smarr formula for smooth horizonless solutions using topology and their application to six-dimensional supergravity with tensor multiplets..

2.1 Komar integrals

Any Killing vector K of a metric on a D -dimensional Lorentzian spacetime defines a conserved quantity through a Komar integral:

$$\mathcal{Q}_K = \frac{1}{8\pi G_D} \int_{\partial V_\infty} \star dK = \frac{1}{8\pi G_D} \int_{\partial V_\infty} (\partial_\mu K_\nu - \partial_\nu K_\mu) d\Sigma^{\mu\nu}, \quad (2.1)$$

where we integrate over a closed spatial surface at infinity. Killing vectors enjoy the property $\nabla^2 K_\mu = -R_{\mu\nu} K^\nu$. With the help of Stokes' theorem, we can then rewrite this as a bulk integral over a volume V on a spatial hypersurface with boundary $\partial V_\infty \cup \partial V_{\text{int}}$:

$$\mathcal{Q}_K = -\frac{1}{4\pi G_D} \int_V \star (K^\mu R_{\mu\nu} dx^\nu) - \frac{1}{8\pi G_D} \int_{\partial V_{\text{int}}} dS^{\mu\nu} (\partial_\mu K_\nu - \partial_\nu K_\mu). \quad (2.2)$$

For a spacetime with a timelike Killing vector K , one usually relates the Komar integral to the ADM mass. However, this is only valid for an energy-momentum tensor that asymptotically approaches that of a weak static dust source, with $T_{00} \gg T_{0i}, T_{ij}$ and $\partial_0 g_{ij} = 0$ asymptotically. For a string-like object spanning the y direction, we expect that T_{00} and T_{yy} will be of the same order, so we need to slightly modify the story.

2.2 ADM integrals

We now review the relevant results of [34]. To relate the Komar integral to physical quantities such as the ADM energy, we consider an energy-momentum tensor that has asymptotically $p+1$ dominating diagonal components $T_{00}, T_{aa}, a = 1 \dots p$ and $p < D-3$. We assume all other components of the energy-momentum tensor are subleading compared to these. We take the p coordinates to be compact and consider the linearization around a flat metric, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with Minkowski reference metric

$$ds_D^2 = -dt^2 + \sum_{a=1}^p dy^a dy^a + \sum_{i=1}^n dx^i dx^i, \quad n = D - p - 1. \quad (2.3)$$

We write the Einstein equations as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_D T_{\mu\nu}. \quad (2.4)$$

The energy density \mathcal{E} , average tension \mathcal{T} and angular momentum density \mathcal{J} are

$$\mathcal{E} = \int d^n x \langle T_{00} \rangle, \quad (2.5)$$

$$\mathcal{T} = -\frac{1}{p} \sum_{a=1}^p \int d^n x \langle T_{aa} \rangle, \quad (2.6)$$

$$\mathcal{J}_{ij} = \int d^n x (x_i \langle T_{j0} \rangle - x_j \langle T_{i0} \rangle). \quad (2.7)$$

with the average over the compact space $\langle X \rangle = 1/V_p \int d^p y X$. From the linearized Einstein equations, one can then deduce the relations to the linearized metric components $h_{\mu\nu}$ [34]:

$$\mathcal{E} = -\frac{1}{16\pi G_D(n-2)} \int_{\partial V_\infty} dS_i \partial_i ((n-1)h_{00} - h_{aa}), \quad (2.8)$$

$$\mathcal{T} = -\frac{1}{p} \frac{1}{16\pi G_D(n-2)} \int_{\partial V_\infty} dS_i \partial_i (p h_{00} - (n+p-2)h_{aa}). \quad (2.9)$$

These are the formulae that relate the asymptotic expansion of an extended object (where T_{aa} is not negligible compared to T_{00}) to its mass and tension. After dimensional reduction over the p internal directions, the ADM mass in $D-p$ dimensions is given by \mathcal{E} . The angular momentum density can still be read off from the off-diagonal metric components:

$$g_{0i} = \frac{16\pi G_D}{\Omega_{D-2}} \frac{x^j J^{ji}}{\rho^n} + \dots, \quad (2.10)$$

where Ω_{D-2} is the volume of the unit $(D-2)$ -sphere and ρ the radius in the four spatial dimensions.

2.3 Normalization of the Komar integrals

We now discuss the relation of the Komar integral to the energy density and tension.

Timelike Killing vector. One readily shows that for a timelike Killing vector K that asymptotes to $K_\infty = \partial_t$, we have the normalization

$$\mathcal{E} - \frac{p}{(D-3)} \mathcal{T} = -\frac{1}{16\pi G_D} \frac{(D-2)}{(D-3)} \int_{\partial V_\infty} dS_{\mu\nu} (\partial^\mu K^\nu - \partial^\nu K^\mu). \quad (2.11)$$

For $p=0$, we retrieve the usual relations between the ADM mass $M = \mathcal{E}$ and the asymptotic form of the metric components [13, 37]

$$g_{00} = -1 + \frac{16\pi G_D}{(D-2)\Omega_{D-2}} \frac{M}{\rho^{D-3}} + \dots, \quad (2.12)$$

$$g_{ij} = \left(1 + \frac{16\pi G_D}{(D-2)(D-3)\Omega_{D-2}} \frac{M}{\rho^{D-3}} \right) \delta_{ij} + \dots \quad (2.13)$$

Null Killing vector. Most of this paper is concerned with supersymmetric solutions in six dimensions. For these, it is useful to discuss $p = 1$ and consider null coordinates:

$$u = \frac{t - y}{\sqrt{2}}, \quad v = \frac{t + y}{\sqrt{2}}. \quad (2.14)$$

For a null Killing vector K that asymptotically becomes $K_\infty = \partial_u$, one finds:

$$\mathcal{E} + \mathcal{T} = -\frac{1}{8\pi G_D} \frac{(n + p - 1)}{(n - 2)} \int_{\partial V_\infty} dS_{\mu\nu} (\partial^\mu K^\nu - \partial^\nu K^\mu). \quad (2.15)$$

Note that these results, as in [34], are in principle only valid for time-independent metric perturbations. Metrics with a null Killing vector ∂_u do not in general have to be time-independent. However, the time-dependence of the metric is heavily constrained. Since we average (integrate) over the internal, compact direction y , the resulting averaged metric must be time-independent and the results for the Komar integrals remain valid.

The normalization of the Komar integral (2.1), which we use in a 6D supergravity context for strings ($p = 1$), implies that:

$$\mathcal{Q}_K = -\frac{1}{2} (\mathcal{E} + \mathcal{T}). \quad (2.16)$$

2.4 Six-dimensional supergravity

Here we discuss the six-dimensional setup relevant for the three-charge black hole. First we consider an arbitrary number n_T of tensor multiplets; for superstrata in six dimensions, $n_T = 2$. We also explicitly give the formulas for $n_T = 1$, which is relevant for all of the examples we discuss except the superstrata of section 3.4.

2.4.1 Minimal supergravity with n_T tensor multiplets

The six-dimensional supergravity theories of relevance to this work have an $SO(n, m)$ global symmetry, with n the number of tensors in the gravity multiplet. In the D1-D5-P frame, the relevant six-dimensional theories are obtained by a compactification on T^4 or K3, which respectively give $\mathcal{N} = (2, 2)$ -supergravity with $SO(5, 5)$ global symmetry and $\mathcal{N} = (2, 0)$ -supergravity with an $SO(5, 21)$ symmetry group.

Luckily, we do not need the full details of these extended supergravity theories. Rather, we can consider a consistent truncation to ‘minimal’ six-dimensional supergravity with only $\mathcal{N} = (1, 0)$ supersymmetry. This theory has $SO(1, n_T)$ global symmetry where n_T is the number of tensor multiplets and is in principle arbitrary as it is unfixed by supersymmetry. For our purposes, n_T will be either 1 or 2, see appendix B for more details on the reduction from 10D. Even though we focus on the theory with $SO(1, n_T)$ global symmetry, our results and in particular the Komar integrals (2.24) and (2.25) below are straightforwardly extended to the bosonic sector of six-dimensional supergravity theories with more supersymmetry, by formally replacing the $SO(1, n_T)$ metric η_{rs} with the metric of the appropriate global symmetry group.

When $n_T > 1$, the equations of motion of the tensor fields do not follow from an action. We can still consider the ‘pseudo-action’ [38, 39] for the bosonic fields⁴

$$\mathcal{L} = \frac{1}{4}R - \frac{1}{2}\partial_\mu v_r \partial^\mu v^s - \frac{1}{12}\mathcal{M}_{rs}G_{\mu\nu\rho}^r G^{s\mu\nu\rho}, \quad (2.17)$$

that captures the equations of motion of the scalar fields and the metric. The scalars parametrize the coset $SO(1, n_T)/SO(n_T)$. They can be organized in the $SO(1, n_T)$ -matrix $V = \begin{pmatrix} v_r \\ x_r^M \end{pmatrix}$ with $M = 1 \dots n_T$ and $r = 0 \dots n_T$.⁵ They enter the tensor dynamics through the scalar metric $\mathcal{M} = \eta V^T V \eta$, with η the $SO(1, n_T)$ -metric, or in index notation

$$\mathcal{M}_{rs} = v_r v_s + x_r^M x_s^M. \quad (2.18)$$

The dynamics of the $n_T + 1$ tensor fields G^r are captured by the self-duality conditions and Bianchi identities

$$\mathcal{M}_{rs}G^s = \eta_{rs} \star G^s, \quad dG^r = 0, \quad (2.19)$$

where \star is the six-dimensional Hodge star operator. Finally, the Einstein equations are:

$$R_{\mu\nu} = 2\partial_\mu v^r \partial_\nu v_r + \frac{1}{2}\mathcal{M}_{rs}G_{\mu\alpha\beta}^r G_\nu^{s\alpha\beta}. \quad (2.20)$$

2.4.2 Smarr Formula

We are concerned with field configurations that respect the symmetry of a Killing vector K . This means the Lie derivative of the fields with respect to K vanishes:

$$\mathcal{L}_K g_{\mu\nu} = 0, \quad \mathcal{L}_K v^r = 0, \quad \mathcal{L}_K G^r = 0. \quad (2.21)$$

Since dG^r and $\mathcal{L}_K = d i_K + i_K d$, we can write the three-form and its dual as

$$i_K G^r = d\Lambda^r + H^r, \quad (2.22)$$

for some globally defined one-forms Λ^r and closed but not exact two-forms H^r . The Einstein equations (2.20) become

$$K^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\rho (\mathcal{M}_{rs}\Lambda_\sigma^r G_\nu^{s\rho\sigma}) + \frac{1}{2}(\mathcal{M}_{rs}H_{\rho\sigma}G_\nu^{s\rho\sigma}). \quad (2.23)$$

Then the Komar integral (2.2) is:

$$\mathcal{Q}_K = -\frac{1}{8\pi G_6} \int_V \mathcal{M}_{rs} H_{\rho\sigma}^r G_\nu^{s\rho\sigma} dV^\nu - \frac{1}{8\pi G_6} \int_{\partial V_{int}} (\mathcal{M}_{rs}\Lambda_\sigma^r G_{\mu\nu}^{s\rho\sigma} dS^{\mu\nu} + (\partial_\mu K_\nu - \partial_\nu K_\mu)). \quad (2.24)$$

As in [13], we find that we can support matter (non-zero Komar integrals) with horizons or with topology. For trivial topology, $H^r = 0$ and the Smarr formula (2.24) relates the

⁴To avoid confusion with standard notation H for harmonic forms, we do not follow the notation of [38, 39] for the three-forms and the kinetic matrix. To convert, use $G^r = H_{theirs}^r$ and $\mathcal{M}_{rs} = (G_{rs})_{theirs}$.

⁵It is customary to write the $SO(1, n_T)$ conditions $V\eta V^T = V^T \eta V = \eta$ in component notation as $v_r v^r = 1, v^r x_r^M = 0, v_r v_s - x_r^M x_s^M = \eta_{rs}$.

Komar integral to horizon quantities (area, charges and angular momenta). If also no horizons are present, the right-hand side of (2.24) is zero and we get a vanishing Komar integral for the Killing vector K .

We are interested in spacetimes without inner boundaries. With (2.19), we find

$$\mathcal{Q}_K = -\frac{1}{4\pi G_6} \int_V \eta_{rs} H^r \wedge G^s, \quad (2.25)$$

so that only non-trivial topology can allow for non-zero Komar integrals.

2.4.3 One tensor multiplet

For many of the solutions in this paper we can restrict to $SO(1,1)$ supergravity with $n_T = 1$. Including only one extra tensor multiplet in addition to the minimal supergravity multiplet is convenient as it allows for a Lagrangian description of the theory. The single self-dual three-form G^+ of the gravity multiplet can be combined with the single anti self-dual three-form G^- of the tensor multiplet in one unconstrained three-form $G = \frac{1}{2}(G^+ + G^-)$. The action becomes

$$\mathcal{L} = \frac{1}{4}R - \frac{1}{2}\partial_\mu X \partial^\mu X - \frac{1}{12}e^{2\sqrt{2}X} G_{\mu\nu\rho} G^{\mu\nu\rho}. \quad (2.26)$$

We introduce the dual three-form (equivalent to (2.19)):

$$\tilde{G} = e^{2\sqrt{2}X} \star G. \quad (2.27)$$

To compare to the discussion of section 2.4.1, we can choose $G^0 = G$, $G^1 = \tilde{G}$. The $SO(1,1)$ metric is then $\eta = \sigma_1$, and one can choose the $SO(1,1)$ scalar matrix as $V = \exp(\sqrt{2}X\sigma_3)$, where σ_i are the Pauli matrices.

The Einstein equation can be (re)written as:

$$R_{\mu\nu} = 2\partial_\mu X \partial_\nu X + \frac{1}{2} \left(e^{2\sqrt{2}X} G_{\mu ab} G_\nu{}^{ab} + e^{-2\sqrt{2}X} \tilde{G}_{\mu ab} \tilde{G}_\nu{}^{ab} \right). \quad (2.28)$$

The Komar integral (2.25) is then

$$\mathcal{Q}_K = -\frac{1}{4\pi G_6} \int_V \left(H \wedge \tilde{G} + \tilde{H} \wedge G \right), \quad (2.29)$$

with the harmonic forms H, \tilde{H} defined through

$$i_K G = d\Lambda + H, \quad \tilde{i}_K \tilde{G} = d\tilde{\Lambda} + \tilde{H} \quad (2.30)$$

for some global one-forms Λ .

2.4.4 Supersymmetry

Let us also mention the fermionic content of the $SO(1,1)$ theory. The gravity multiplet consists of $(e_\mu, \psi_\mu^\alpha, B_{\mu\nu}^+)$ with B^+ a self-dual tensor such that $G^+ \equiv dB^+ = \star G^+$. The tensor multiplet consists of $(B_{\mu\nu}^-, \chi^\alpha, X)$ with $G^- \equiv dB^- = -\star G^-$. The supersymmetry transformations of the fermions are:

$$\delta\psi_\mu^\alpha = \left(\partial_\mu - \frac{1}{4}e^{\sqrt{2}X} G_{\mu\nu\sigma}^+ \gamma^{\nu\sigma} \right) \varepsilon^\alpha, \quad (2.31)$$

$$\delta\chi^\alpha = \frac{1}{2i} (\sqrt{2}\gamma^\mu \partial_\mu X + \frac{1}{6}e^{\sqrt{2}X} G_{\mu\nu\rho}^- \gamma^{\mu\nu\rho}) \varepsilon^\alpha. \quad (2.32)$$

Given a Killing spinor ϵ^α we can construct the bilinear vector:

$$K_\mu \epsilon^{\alpha\beta} = \bar{\epsilon}^\alpha \gamma_\mu \epsilon^\beta, \quad (2.33)$$

which is always a *null* Killing vector, $K \cdot K = 0$. The supersymmetry equations imply (using the form notation $K \equiv K^\mu g_{\mu\nu} dx^\nu$):

$$dK = 2e^{\sqrt{2}X} i_K G^+ = i_K (e^{\sqrt{2}X} G + e^{-\sqrt{2}X} \tilde{G}), \quad (2.34)$$

$$i_K dX = 0, \quad (2.35)$$

since the self-dual part of G is given by $G^+ = 1/2(G + e^{-2\sqrt{2}X} \tilde{G})$. Using $i_K \star G = \star(G \wedge K)$, this allows us to write the null charge associated with K as

$$\mathcal{Q}_K = \frac{1}{8\pi G_6} \int_{\partial V_\infty} \star dK = -\frac{1}{8\pi G_6} \int_{\partial V_\infty} (\tilde{G} + G) \wedge K, \quad (2.36)$$

where we have assumed that $X = 0$ at infinity, which we can always do for asymptotically flat spacetimes. In the microstate geometries of section 3, we find that $\partial V_\infty = S^1 \times S^3$, and the Killing vector K projected on this spacelike surface is (proportional to) the isometry along the compact S^1 . In the notation of the metric (3.1) below, $K = -dv$ at spatial infinity. This means we simply get:

$$\mathcal{Q}_K = -\frac{L_v}{8\pi G_6} \int_{S^3} (\tilde{G} + G) = -\frac{L_v \pi}{4G_6} (Q_e + Q_m), \quad (2.37)$$

where $L_v = 2\pi R_v$ is the size of the S^1 direction parametrized by v (at constant time). This relation is thus simply the BPS condition in 6D relating the null charge associated to K to the electric and magnetic charges of the solution.

3 Supersymmetric Examples

We now analyze in detail the null Komar integral for known smooth supersymmetric solutions to six-dimensional supergravity. The structure of supersymmetric solutions in 6D minimal supergravity was studied in [35] and including an additional vector multiplet and one tensor multiplet in [40]. Using the Killing spinors of such supersymmetric solutions, one can always construct a null Killing vector which locally is $V = \partial_u$. The metric can then be shown to take the form:

$$ds_6^2 = -2H^{-1}(dv + \beta_i dx^i)[du + \omega_i dx^i + \frac{\mathcal{F}}{2}(dv + \beta_i dx^i)] + H dx_4^2, \quad (3.1)$$

where dx_4 is the line element on the 4D “base space” \mathcal{B} , the one-forms $\beta = \beta_i dx^i$, $\omega = \omega_i dx^i$ only have legs on \mathcal{B} and the functions $H, \beta_i, \omega_i, \mathcal{F}$ are in general functions of v and all of the 4D base coordinates x^i . The conditions that these functions (and the three-form and scalar) must satisfy for supersymmetric solutions can be found in [40], or [41] whose conventions and notation we follow. Note that the ansatz (3.1) only holds for sections 3.1-3.3, in section 3.4 we extend the ansatz for two tensor multiplets.

3.1 General expectations

It is instructive to first work out the ADM integrals \mathcal{E} and \mathcal{T} for the three-charge solutions of our interest. Asymptotically, the metric (3.1) approaches that of the three-charge black string for which $H = (Z_2 Z_3)^{-1/2}$, $\mathcal{F} = -Z_1$, $\omega = 0$, $\beta = 0$ and $Z_i = 1 + Q_i/\rho^2$, with ρ the standard radial coordinate of the 4D base $\mathcal{B} = \mathbb{R}^4$. The asymptotic metric perturbation in the coordinates t, y (2.14) is

$$h_{00} = \frac{1}{2} \frac{Q_2 + Q_3 + Q_1}{\rho^2} + O(\rho^3), \quad h_{yy} = \frac{1}{2} \frac{-Q_2 - Q_3 + Q_1}{\rho^2} + O(\rho^3). \quad (3.2)$$

and we find that

$$\mathcal{E} = \frac{\pi L_y}{4\pi G_6} \left(Q_2 + Q_3 + \frac{1}{2} Q_1 \right), \quad \mathcal{T} = \frac{\pi L_y}{4\pi G_6} \left(Q_2 + Q_3 - \frac{1}{2} Q_1 \right), \quad (3.3)$$

with $y \sim y + L_y$. Note that \mathcal{E} is the ADM mass after dimensional reduction over the y -circle.⁶ Using (2.16), we anticipate that the Komar integral will be:

$$\mathcal{Q}_K = -\frac{1}{2}(\mathcal{E} + \mathcal{T}) = -\frac{\pi L_y}{4G_6}(Q_2 + Q_3), \quad (3.4)$$

and does not involve the momentum charge Q_1 .

3.2 The uplift of five-dimensional microstate geometries

As a warm-up, we consider the uplift of five-dimensional microstate geometries. Komar integrals and Smarr formulae for those geometries were discussed at length in [13], hence we do not go into much detail here. The solutions are completely smooth multi-centered solutions of the 5D STU model with three gauge fields A^I ($I = \{1, 2, 3\}$) and three scalars X^I , constrained by $X^1 X^2 X^3 = 1$. The 5D Lagrangian is given by (A.11). The 6D theory of minimal supergravity coupled to one tensor multiplet (2.26) gives exactly this STU model when dimensionally reduced to 5D. See appendix A for more details.

The 5D solutions that we are interested in are given by the metric [42–44]:

$$ds_5^2 = -Z^{-2}(dt + k)^2 + Z ds_4^2, \quad Z = (Z_1 Z_2 Z_3)^{1/3}. \quad (3.5)$$

where the 4D base space \mathcal{B} is Gibbons-Hawking: it is a $U(1)$ fibration with coordinate ψ over flat \mathbb{R}^3 . The solutions are then determined by specifying the poles of eight functions V, K^I, L_I, M , which are harmonic functions on \mathbb{R}^3 . For instance, we have $Z_I = L_I + C_{IJK} K^J K^K / 2V$ with $C_{IJK} = |\epsilon_{IJK}|$. These eight harmonic functions must satisfy stringent conditions in order for the full 5D spacetime to be completely regular and asymptotically flat [1, 13].

The gauge potentials in 5D are:

$$A^I = -Z_I^{-1}(dt + k) + B^I, \quad (3.6)$$

⁶Note that the dimensional reduction in section 3.2 and appendix A.2 is a reduction over the spacelike v -circle, which will give a different resulting 5D ADM mass in terms of Q_1 , see eq. (3.10).

where B^I is a magnetic potential (only well-defined locally). The scalars are given by:

$$X^I = \frac{Z}{Z_I}. \quad (3.7)$$

For asymptotically flat 5D spacetimes, we have asymptotically:

$$Z_I \sim 1 + \frac{Q_I}{4r} = 1 + \frac{Q_I}{\rho^2}, \quad (3.8)$$

with r the usual radial coordinate on \mathbb{R}^3 and $\rho = 4r$ is the radial coordinate on the four-dimensional base. In microstate geometry literature, the charges Q_I are normalized through the asymptotic expansion of the electric field in 5D as $F_{0\rho} \sim 2\frac{Q_I}{\rho^3}$ and not with factors involving the volume of the three sphere that are more common from Gaussian integrals. This means that we have:

$$-\frac{1}{16\pi G_5} \int_{\partial V_\infty} \star_5 F_I = \frac{\pi}{4G_5} Q_I. \quad (3.9)$$

For the six-dimensional metric, scalar and tensor solutions see eqs. (A.20).

3.2.1 The topology of the base

The poles of V (‘centers’) indicate where the ψ -fibre degenerates in the 4D base space (although the complete 5D spacetime is always completely smooth). Since the ψ -fibre degenerates at each center, we can construct non-contractible compact two-cycles in the 4D space, which are also compact two-cycles in the full 5D geometry. These two-cycles are constructed by taking the ψ -fibration over an arbitrary path in \mathbb{R}^3 between two centers. This completely determines the 5D homology structure of simply connected solutions. For $N = 2p + 1$ centers, the global topology is that of a p -fold connected sum of $(S^2 \times S^2)$ with a point removed, for $N = 2p$ centers the topology is $(\mathbb{R}^2 \times S^2) \# (S^2 \times S^2) \# \dots \# (S^2 \times S^2)$.⁷

The five-dimensional ADM mass of these solutions can be written as [13]

$$M_{ADM,5D} = -\frac{1}{32\pi G_5} C_{IJK} \alpha^I \int_{\Sigma_4} F^J \wedge F^K = \frac{\pi}{4G_5} \alpha^I Q_I = \frac{\pi}{4G_5} (Q_1 + Q_2 + Q_3), \quad (3.10)$$

where $\alpha^I = 1$ for asymptotically flat solutions and Σ_4 is a spacelike surface of constant time. The integral of $F^J \wedge F^K$ is computed “entirely with cohomology”, by calculating the flux of the F^I over the non-trivial compact two-cycles of the geometry as well as the intersection number of these two-cycles.

3.2.2 The topology of the uplift

The six-dimensional uplift of (3.5) is a non-trivial fibration of the new coordinate v . From the expression for the three-form:

$$2G = (X^3)^{-2} \star_5 F^3 + F^2 \wedge (dv + A^1), \quad (3.11)$$

⁷We only discuss $V = \sum_i q_i/|x - x_i|$ with $|q_i| = 1$, such that the centers are smooth points in the full space, and $\sum_i q_i = 1$, such that the space is asymptotically flat.

we can easily see that we have:

$$2i_K G = d(\lambda_2(dv + A^1)) + d(Z_1^{-1}Z_2^{-1}(dt + k)) + F^1, \quad (3.12)$$

where we have defined $\lambda_I = Z_I^{-1} - 1$. The form given in the first term, $\lambda_2(dv + A^1)$, is well-defined. The second term is $Z_1^{-1}Z_2^{-1}(dt + k)$ and is also a well-defined form (as discussed in [13]). This implies the cohomology split:

$$2\Lambda = \lambda_2(dv + A^1) + Z_1^{-1}Z_2^{-1}(dt + k), \quad (3.13)$$

$$2H = F^1. \quad (3.14)$$

Similarly, we can find $\tilde{\Lambda}, \tilde{H}$ by switching the roles of Z_2 and Z_3 in the above expressions. Note that also $2\tilde{H} = F^1$.

The null charge is then:

$$\mathcal{Q}_K = -\frac{1}{4\pi G_6} \int_V (H \wedge \tilde{G} + \tilde{H} \wedge G) \quad (3.15)$$

$$= -\frac{1}{16\pi G_6} \int_V (F^1 \wedge (F^3 \wedge dv) + F^1 \wedge (F^2 \wedge dv)) \quad (3.16)$$

$$= \frac{L_v}{16\pi G_6} \int_{\Sigma_4} (F^1 \wedge F^3 + F^1 \wedge F^2) \quad (3.17)$$

$$= -\frac{L_v \pi}{4G_6} (Q_2 + Q_3), \quad (3.18)$$

where we used the cohomological computation of the integral $F^I \wedge F^J$ in 5D over Σ_4 [13], and $V = S^1(v) \times \Sigma_4$. We see that the null charge is simply the sum of electric and magnetic (string) charges. Note that in five dimensions, Q_1 is on the same footing as $Q_{2,3}$, but in six dimensions it is a momentum charge and does not appear in the null charge \mathcal{Q}_K .

The analysis above shows us that we clearly still have non-trivial compact two-cycles in six dimensions which are given by the trivial uplift of the two-cycles of the five-dimensional solution. These are the cycles supporting the cohomological flux $H, \tilde{H} \sim F^1$. The S^1 -fibration of the coordinate v over the compact two-cycles of the five-dimensional geometry also introduces new non-trivial three-cycles. Over these cycles, the cohomology elements $F^{2,3} \wedge dv$ have non-zero flux.

However, this is not quite the end of the story. In 6D, we must also have a non-trivial three-sphere at infinity. Indeed, the (electric string) charge in 6D is defined as:

$$Q_e = \frac{1}{2\pi^2} \int_{S^3(\infty)} e^{2\sqrt{2}X} \star G, \quad (3.19)$$

where S^3 is the S^3 at infinity perpendicular to the string which is along v . Since the equation of motion for the three-form is simply $d(e^{2\sqrt{2}X} \star G) = 0$, this S^3 at infinity must be non-contractible to be able to support non-zero flux for smooth solutions free of singularities. Note that this non-trivial three-cycle is absent in the original 5D geometry. This can be explained by the fact that this three-cycle must be homologically equivalent to an $S^1(v)$ fibration over a two-cycle in the 4D base (which we mentioned above). These new (compared to 5D) non-trivial three-cycles in constant time-slices of the six-dimensional geometry are an interesting feature of the $S^1(v)$ uplift.

3.3 D1-D5 microstate geometries and supertubes

We are now ready to discuss the topology and the Komar integral for more generic solutions of the D1-D5-P system. In this section, we first focus on the D1-D5 supertube solutions of Lunin and Mathur [16–18]. As we explain in section 3.4, the result (3.39) for the Komar integral is the same for more generic D1-D5 supertubes and D1-D5-P superstrata, since those describe wiggles of the D1-D5 supertube and are topologically equivalent.

The D1-D5 Lunin-Mathur geometries are solutions to six-dimensional supergravity with only one tensor multiplet:

$$ds^2 = -\frac{2}{\sqrt{Z_1 Z_2}}(dv + \beta)(du + \omega) + \sqrt{Z_1 Z_2} ds_4^2, \quad (3.20)$$

$$e^{2\sqrt{2}X} = \frac{Z_1}{Z_2}, \quad (3.21)$$

$$2B = -Z_1^{-1}(du + \omega) \wedge (dv + \beta) + \gamma_2. \quad (3.22)$$

Here ds_4^2 is the 4D flat metric with coordinates x_i ($i = \{1, \dots, 4\}$) and $a_1, \gamma_2, \beta, \omega$ are forms on the 4-manifold. The D1-D5 microstate is completely determined by profile functions $g_i(v), i = 1 \dots 4$ with $0 \leq v \leq L$. Certain important functions are given by (for the complete list of fields, see for example [22]):

$$Z_2 = 1 + \frac{Q_5}{L} \int_0^L \frac{1}{|x_i - g_i(v')|^2} dv', \quad Z_1 = 1 + \frac{Q_5}{L} \int_0^L \frac{|\dot{g}_i(v')|^2}{|x_i - g_i(v')|^2} dv', \quad (3.23)$$

$$A = -\frac{Q_5}{L} \int_0^L \frac{\dot{g}_j(v') dx^j}{|x_i - g_i(v')|^2} dv', \quad dB = -\star_4 dA, \quad (3.24)$$

$$\beta = \frac{-A + B}{\sqrt{2}}, \quad \omega = \frac{-A - B}{\sqrt{2}}, \quad (3.25)$$

$$d\gamma_2 = \star_4 dZ_2. \quad (3.26)$$

Perhaps the easiest explicit profile is the once-wound circle, given by (with $L = 2\pi R_y$):

$$g_1(v) = a \cos(v/R_y), \quad g_2(v) = a \sin(v/R_y), \quad g_3(v) = g_4(v) = 0. \quad (3.27)$$

Then we can parametrize the (flat) 4D metric as:

$$ds_4^2 = \frac{f}{r^2 + a^2} dr^2 + f d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2 + r^2 \cos^2 \theta d\psi^2, \quad (3.28)$$

and the above functions become:

$$Z_1 = 1 + \frac{Q_1}{f}, \quad Z_2 = 1 + \frac{Q_5}{f}, \quad (3.29)$$

$$A = -a\sqrt{Q_1 Q_5} \frac{\sin^2 \theta}{f} d\phi, \quad B = -a\sqrt{Q_1 Q_5} \frac{\cos^2 \theta}{f} d\psi, \quad (3.30)$$

$$f = r^2 + a^2 \cos^2 \theta, \quad (3.31)$$

where $Q_1 = a^2 R_y^2 / Q_5$, and the D1-D5 string at $x_i = F_i(v)$ is now at $r = 0, \theta = \pi/2$ ($f = 0$).

3.3.1 Topology and homology

The topology of the D1-D5 system with once-wound circular profile was discussed in [18]. Any D1-D5 geometry with profile $g'_i(v)$ that can be continuously deformed into a circle will share the same topology of $\mathbb{R}^2 \times S^3$. At infinity we have an $S^3(\theta, \phi, \psi)$ of the 4D base, which deforms continuously to the non-trivial $S^3(\theta, \tilde{\phi}, \tilde{\psi})$ in the interior with $\tilde{\phi} = \phi + t/R$, $\tilde{\psi} = \psi + y/R$, while $S^1(y)$ (keeping $\tilde{\psi}$ fixed) shrinks to zero size in the interior.

Hence we clearly have exactly one non-trivial three-cycle given by the three-sphere at infinity, and one non-trivial (non-compact) two cycle, given by the volume element of the \mathbb{R}^2 factor. The three-cycle is again needed in this singularity-free geometry in order for the geometry to be able to support non-zero three-form flux. The intersection number between the two-cycle and the three-cycle is simply +1 (with suitable orientations of the cycles).

3.3.2 Cohomology and null charge

For a general D1-D5 geometry, we have:

$$2 i_k G = d(Z_1^{-1}(dv + \beta)) \quad (3.32)$$

$$= \frac{1}{\sqrt{2}} d(Z_1^{-1}(dy + B) + Z_1^{-1}(dt - A)). \quad (3.33)$$

Note that there is no obvious easy split to be made by defining $\lambda_1 = Z_1^{-1} - 1$ and splitting off terms proportional to λ_1 . This is because the fibres A, B typically have singularities on the string profile and/or in the origin. So we can leave the well-behaved one-form Λ implicit:

$$2 H \equiv i_K G - d\Lambda = d(Z_1^{-1}(dv + \beta)) - d\Lambda, \quad (3.34)$$

since the integrals we will perform are independent of Λ anyway. In the explicit example of the once-wound circular profile, we can easily see that

$$\frac{1}{L_v} \int_{\mathbb{R}^2} H = \frac{1}{L_v} \left(\frac{L_v}{2} \right) = \frac{1}{2}, \quad (3.35)$$

where we integrate the \mathbb{R}^2 cycle from the string profile (at $r = 0, \theta = \pi/2$) to $r = \infty$, and we used that $Z_1^{-1}(f = 0) = 0$ and $Z_1^{-1}(r = \infty) = 1$.

We see that H is the cohomological dual of the non-trivial two-cycle in the geometry, as expected. The harmonic part of the three-form G and its dual \tilde{G} are both proportional to the volume form of the non-trivial three-cycle S^3 :

$$\frac{1}{2\pi^2} \int_{S^3(\infty)} G = Q_5, \quad \frac{1}{2\pi^2} \int_{S^3(\infty)} \tilde{G} = Q_1, \quad (3.36)$$

as these parts precisely define the D1 and D5 charges of the geometry. Putting this together

gives for the null charge:

$$\mathcal{Q}_K = -\frac{1}{4\pi G_6} \int_{\mathbb{R}^2 \times S^3} (H \wedge \tilde{G} + \tilde{H} \wedge G) \quad (3.37)$$

$$= -\frac{1}{4\pi G_6} \left(\int_{\mathbb{R}^2} H \right) (+1) \left(\int_{S^3} \tilde{G} \right) - \frac{1}{4\pi G_6} \left(\int_{\mathbb{R}^2} \tilde{H} \right) (+1) \left(\int_{S^3} G \right) \quad (3.38)$$

$$= -\frac{L_v \pi}{4G_6} (Q_1 + Q_5), \quad (3.39)$$

where we used the intersection number to split the integral into separate integrals over the non-trivial cycles.

3.4 D1-D5-P superstrata

The most general three-charge microstate geometries that fall within six-dimensional supergravity arise from reduction on a rigid T^4 [22]. These solutions excite all IIB supergravity fields in ten dimensions (metric, Ramond-Ramond fields $C_{(0)}, C_{(2)}, C_{(4)}$, as well as $B_{(2)}$ and the dilaton ϕ_1). The solutions can be interpreted as solutions in minimal supergravity in six dimensions coupled to *two* tensor multiplets, see appendix B.

These solutions require extending the results of section 3.3 in two ways: considering an extra tensor multiplet, and adding the momentum charge P. Only then can we cover both generic D1-D5 geometries with a rigid T^4 [19] and the D1-D5-P superstrata [31]. However, these more general solutions are topologically equivalent to the D1-D5 supertubes (3.20). We will show that the Komar integral is unchanged.

The general superstrata solutions as given in [25, 31], in six-dimensional language, fit within the ansatz [22, 31]:

$$ds^2 = \frac{\mathcal{P}}{Z_1 Z_2} \left(-\frac{2}{\sqrt{\mathcal{P}}} (dv + \beta) \left[du + \omega + \frac{\mathcal{F}}{2} (dv + \beta) \right] + \sqrt{\mathcal{P}} ds_4^2 \right), \quad (3.40)$$

$$e^{2\phi} = \frac{Z_1^2}{\mathcal{P}}, \quad (3.41)$$

$$\chi = \frac{Z_4}{Z_1}, \quad (3.42)$$

$$2B = -\frac{Z_2}{\mathcal{P}} (du + \omega) \wedge (dv + \beta) + a_1 \wedge (dv + \beta) + \gamma_2, \quad (3.43)$$

$$B' = -\frac{Z_4}{\mathcal{P}} (du + \omega) \wedge (dv + \beta) + a_4 \wedge (dv + \beta) + \delta_2, \quad (3.44)$$

$$\mathcal{P} = Z_1 Z_2 - Z_4^2, \quad (3.45)$$

where, similar to the D1-D5 ansatz, ds_4^2 is the 4D flat metric and $\beta, \omega, a_1, a_4, \gamma_2, \delta_2$ are forms on this 4D base. We refer to [22, 31] for the full set of supersymmetry equations and equations of motion and only quote those that we need:

$$d\gamma_2 = \star_4 dZ_2, \quad d\delta_2 = \star_4 dZ_4. \quad (3.46)$$

The tensor B comes from the dimensional reduction of $C_{(2)}$ while B' descends from $B_{(2)}$ in 10D; the scalar ϕ is simply the 10D dilaton while χ is the 10D axion $C_{(0)}$. For

more information on the dimensional reduction from 10D to 6D and the realization of the $SO(1, 2)$ symmetry, see appendix B. This ansatz reduces to the D1-D5 ansatz (3.20) when $Z_4 = a_4 = \delta_2 = 0$; the tensor multiplet parametrized by the fields B', χ is set to zero, truncating the $SO(1, 2)$ theory down to $SO(1, 1)$.

The tensor multiplet scalars $\tau = \chi + ie^{-\phi}$ parametrize the coset $SO(1, 2)/SO(2)$. While B and its field strength $G = dB$ are unconstrained, the tensor B' satisfies a duality relation. Indeed, the field strength:

$$G' = dB' - 2\frac{\chi}{e^{-2\phi} + \chi^2}dB, \quad (3.47)$$

is anti self-dual in six dimensions:

$$G' = -\star G'. \quad (3.48)$$

Thus, we find the correct tensor field content for the $SO(1, 2)$ theory of minimal supergravity with two tensor multiplets.

The null charge is given by (see also appendix B):

$$\mathcal{Q}_K = -\frac{1}{4\pi G_6} \int_V \left(H \wedge \tilde{G} + \tilde{H} \wedge G \right) + \frac{1}{8\pi G_6} \int_V (H' \wedge G'), \quad (3.49)$$

where H, \tilde{H} are defined as in (2.30), similarly H' is the harmonic part of $i_K G'$, and the dual form \tilde{G} is now defined by:

$$\tilde{G} = \frac{e^{2\phi}}{1 + e^{2\phi}\chi^2} \star G. \quad (3.50)$$

For the superstrata of [31], the terms in (3.49) involving G, \tilde{G} can easily be seen to give the same contribution $\sim (Q_1 + Q_5)$ as for the D1-D5 microstates above. The term involving G' does not contribute. It is easiest to realize this by seeing that dB' and χdB fall off too fast at infinity to have a non-zero integral $\int_{S^3_\infty} G'$; in essence, this is because Z_4 falls off faster at infinity than Z_1 or Z_2 (which give the Q_1, Q_5 contributions to the null charge as

in the D1-D5 case above).⁸ We conclude that:

$$\mathcal{Q}_K = -\frac{L_v \pi}{4G_6} (Q_1 + Q_5), \quad (3.54)$$

just as for the D1-D5 supertube.

That the null charge gives the same result for D1-D5-P superstrata as for the D1-D5 supertubes is not so surprising from a topological point of view. The important thing to note is that a generic superstratum solution has the same topology as the D1-D5 round supertube. Superstrata describe fluctuations on top of a topologically non-trivial S^3 (shape modes depending on two variables), just as generic two-charge supertubes describe one-dimensional shape modes on the S^3 . This is the same S^3 present for the round supertube discussed in section 3.3, and therefore supertubes and superstrata have a similar topological three-cycle.

4 Non-Extremal Example

We now discuss the JMaRT solutions of [36], which have an interpretation as microstate geometries of the five-dimensional overspinning three-charge black hole. In the IIB frame, these are smooth solitons, with a natural interpretation in six-dimensional supergravity after dimensional reduction on the compact T^4 .

4.1 Metric and gauge fields

The solitons are obtained by demanding the metric ansatz appropriate for describing the non-extremal three-charge black hole to be smooth. Usually, the five-dimensional physical charges are quoted, which in this case are the ADM mass $M_{ADM,5D}$, the electric charges

⁸To see this fall-off explicitly we quote the behaviour for the most general D1-D5 supertube invariant under T^4 rotations. This has five profile components $g_i, i = 1 \dots 4$ and g_5 , and the fields are [45]:

$$\begin{aligned} Z_2 &= 1 + \frac{Q_5}{L} \int_0^L \frac{1}{|x_i - g_i(v')|^2} dv', & Z_4 &= -\frac{Q_5}{L} \int_0^L \frac{\dot{g}_5(v')}{|x_i - g_i(v')|^2} dv', \\ Z_1 &= 1 + \frac{Q_5}{L} \int_0^L \frac{|\dot{g}_i(v')|^2 + |\dot{g}_5(v')|^2}{|x_i - g_i(v')|^2} dv', & d\gamma_2 &= *_4 dZ_2 \quad d\delta_2 = *_4 dZ_4, \\ A &= -\frac{Q_5}{L} \int_0^L \frac{\dot{g}_j(v') dx^j}{|x_i - g_i(v')|^2} dv', & dB &= - *_4 dA, \\ \beta &= \frac{-A+B}{\sqrt{2}} & \omega &= \frac{-A-B}{\sqrt{2}} \quad \mathcal{F} = 0, \quad a_1 = a_4 = x_3 = 0, \end{aligned} \quad (3.51)$$

An explicit example is a round profile in the \mathbb{R}^4 base and a non-zero g_5 component:

$$g_1(v) = a \cos(v/R_y), \quad g_2(v) = a \sin(v/R_y), \quad g_3(v) = g_4(v) = 0, \quad g_5(v) = -\frac{b}{k} \sin(v/R_y). \quad (3.52)$$

The D1-D5 seed solution of [31] starts from such a profile. Then we have that

$$Z_1 = 1 + \frac{Q_1}{f} + c_1 \frac{\sin^{2k} \theta \cos(2k\phi)}{(r^2 + a^2)^k f}, \quad Z_2 = 1 + \frac{Q_5}{f}, \quad Z_4 = c_4 \frac{\sin^k \theta \cos(k\phi)}{\sqrt{r^2 + a^2} f}, \quad (3.53)$$

where $c_1 = \frac{Q_1 a^2 b^2}{2a^2 + b^2}$ and $c_4 = \sqrt{\frac{Q_1 Q_5}{a+2+b^2/2}} b a^k$ are constants. Clearly Z_4 falls off too fast for the $H' \wedge G'$ -term to contribute to the Komar integral. For superstrata solutions, we refer to [25, 31].

Q_1, Q_5, Q_p , and the two angular momenta J_ψ, J_ϕ :⁹

$$M_{ADM,5D} = \frac{L_y \pi}{4G_6} \frac{m}{2} \sum_I \cosh 2\delta_I, \quad J_\psi = -\frac{L_y \pi}{4G_6} m(a_1 c_1 c_2 c_3 - a_2 s_1 s_2 s_3), \quad (4.1)$$

$$Q_I = \frac{m}{2} \sinh 2\delta_I, \quad J_\phi = -\frac{L_y \pi}{4G_6} m(a_2 c_1 c_2 c_3 - a_1 s_1 s_2 s_3), \quad (4.2)$$

given in terms of parameters $m, \delta_1, \delta_5, \delta_p, a_1, a_2$ and with the notation $s_i = \sinh \delta_i, c_i = \cosh \delta_i$. The supersymmetric limit is $m, a_1, a_2 \rightarrow 0, \delta_i \rightarrow \infty$ while keeping $Q_I, m/\sqrt{a_i}$ fixed. We note that the 6D ADM mass (for the asymptotically $\mathbb{R}^{4,1} \times S^1$ spacetime) is actually:

$$M_{ADM,6D} = \frac{L_y \pi}{4G_6} \frac{m}{2} (\cosh 2\delta_1 + \cosh 2\delta_5 + 2 \cosh 2\delta_p), \quad (4.3)$$

so the contribution due to the momentum charge (which is the charge from the graviphoton in reducing from 6D to 5D) is different.

We choose to write the metric and gauge fields in the notation of [46]. The metric, scalar and gauge field in 6D are (note that $B = -C_2/2$, with C_2 the RR two-form of [36]):

$$ds_6^2 = \frac{1}{H_p(H_1 H_5)^{1/2}} \left[-H_m (dt + k)^2 + H_p^2 \left((dy + B_p^m + \frac{c_p}{s_p} k) + \frac{c_p}{s_p} (H_p^{-1} - 1)(dt + k) \right)^2 \right] + (H_1 H_5)^{1/2} ds_4^2, \quad (4.4)$$

$$e^{2\sqrt{2}X} = \frac{H_1}{H_5}, \quad (4.5)$$

$$\begin{aligned} -2B &= \frac{c_1}{s_1} dt \wedge dy - \frac{c_1}{s_1} H_1^{-1} (dt + k) \wedge dy - B_1 \wedge dz - \frac{c_1 c_p}{s_1 s_p} H_1^{-1} dt \wedge dk \\ &\quad - \frac{s_p}{c_p} dt \wedge B_1 - \frac{c_1}{s_1} H_1^{-1} dt \wedge B_3 + m s_5 c_5 \frac{r^2 + a_2^2 + m s_1^2}{f H_1} \cos^2 \theta d\psi \wedge d\phi. \end{aligned} \quad (4.6)$$

where the quantities used are defined by:

$$\begin{aligned} ds_4^2 &= f \left(\frac{r^2}{g} dr^2 + d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\psi^2 \right) \\ &\quad + H_m^{-1} (a_1 \cos^2 \theta d\psi + a_2 \sin^2 \theta d\phi)^2 - (a_2 \cos^2 \theta d\psi + a_1 \sin^2 \theta d\phi)^2, \end{aligned} \quad (4.7)$$

$$\begin{aligned} k &= \frac{m}{f} \left[-\frac{c_1 c_5 c_p}{H_m} (a_1 \cos^2 \theta d\psi + a_2 \sin^2 \theta d\phi) + s_1 s_5 s_p (a_2 \cos^2 \theta d\psi + a_1 \sin^2 \theta d\phi) \right], \\ B^{(i)} &= \frac{m}{f H_m} \frac{c_1 c_5 c_p}{s_I c_I} (a_1 \cos^2 \theta d\psi + a_2 \sin^2 \theta d\phi). \end{aligned} \quad (4.8)$$

Everything is built from the following functions:

$$\begin{aligned} H_i &= 1 + \frac{m s_i^2}{f}, & H_m &= 1 - \frac{m}{f}, \\ f &= r^2 + a_1 \sin^2 \theta + a_2^2 \cos^2 \theta, & g &= (r^2 + a_1^2)(r^2 + a_2^2) - m r^2 = (r^2 - r_+^2)(r^2 - r_-^2), \end{aligned} \quad (4.9)$$

⁹Standard conventions in the literature are to take $G_5 = \pi/4$, which would render the prefactor $L_y \pi / (4G_6) = 1$. As in the rest of the paper, we instead choose to keep the explicit factors of G_6 in all of the relevant formulae. We also choose a normalization for the Q_I that is the same as the rest of the paper, instead of the usual normalization which would include a factor of $L_y \pi / (4G_6)$ in the definition of the Q_I as well.

The three-form is simply $G = dB$. The dual potential, $\tilde{G} = d\tilde{B}$ is then given by:

$$\tilde{B} = B \text{ with } s_1 \leftrightarrow s_5; c_1 \leftrightarrow c_5; H_1 \leftrightarrow H_5. \quad (4.10)$$

4.2 Constraints for smooth solutions

Smooth JMaRT solutions are determined for fixed charges Q_1, Q_5, Q_p , by two integers m, n . One can extend these to include \mathbb{Z}_k orbifolds with k an integer. They have the following relations between their parameters:

$$r_+^2 = -a_1 a_2 \frac{s_1 s_5 s_p}{c_1 c_5 c_p}, \quad (4.11)$$

$$M = a_1^2 + a_2^2 - a_1 a_2 \left[\frac{c_1^2 c_5^2 c_p^2 + s_1^2 s_5^2 s_p^2}{s_1 c_1 s_5 c_5 s_p c_p} \right]. \quad (4.12)$$

The constant t slices have the topology of $\mathbb{R}^2 \times S^3 / \mathbb{Z}_k$. The non-contractible S^3 is spanned at the origin $r = r_+$ by the coordinates $\theta, \tilde{\psi}, \tilde{\phi}$, with the identifications

$$\tilde{\psi} = \psi - \frac{s_p c_p}{a_2 c_1 c_5 c_p - a_1 s_1 s_5 s_p} y, \quad \tilde{\phi} = \phi - \frac{s_p c_p}{a_1 c_1 c_5 c_p - a_2 s_1 s_5 s_p} y, \quad (4.13)$$

The following quantization conditions ensure that the identification $y \rightarrow y + 2\pi R$ is a closed orbit:

$$\frac{s_p c_p}{a_2 c_1 c_5 c_p - a_1 s_1 s_5 s_p} R = m, \quad \frac{s_p c_p}{a_1 c_1 c_5 c_p - a_2 s_1 s_5 s_p} R = n, \quad (4.14)$$

for integers m, n .

The \mathbb{R}^2 factor has a smooth origin at $r = r_+$, where the $t = \text{constant}$ part of the metric has the form (up to irrelevant constant prefactors)

$$ds^2|_{dt=0} = d\rho^2 + \frac{\rho^2}{R^2} dy^2, \quad (4.15)$$

with the identification $y \sim y + 2\pi Rk$ and the radius given by

$$R = \frac{M s_1 c_1}{\sqrt{a_1 a_2}} \frac{\sqrt{s_1 c_1 s_5 c_5 s_p c_p}}{c_1^2 c_5^2 c_p^2 - s_1^2 s_5^2 s_p^2}. \quad (4.16)$$

4.3 Komar integral

We want to study the Komar integral, which reduces for this topology to

$$\mathcal{Q}_K = -\frac{1}{4\pi G_6} \int_V (H \wedge \tilde{G} + \tilde{H} \wedge G) = -\frac{1}{4\pi G_6} \left(\int_{\mathbb{R}^2} H \int_{S^3} \tilde{G} + \int_{\mathbb{R}^2} \tilde{H} \int_{S^3} G \right) \quad (4.17)$$

The non-contractible S^3 is homologically equivalent to the one at infinity appearing in Gauss' law. Hence we can perform the S^3 integral at spatial infinity:

$$\frac{1}{4\pi G_6} \int_{S^3(\infty)} G = -\frac{1}{8\pi G_6} \lim_{r \rightarrow \infty} \int d \left[m s_5 c_5 \frac{r^2 + a_2^2 + m s_1^2}{f H_1} \cos^2 \theta d\psi \wedge d\phi \right] \quad (4.18)$$

$$= -\frac{\pi}{4G_6} \lim_{r \rightarrow \infty} m s_5 c_5 \frac{r^2 + a_2^2 + m s_1^2}{f H_1} \cos^2 \theta \Big|_{\theta=0}^{\theta=\pi/2} = \frac{\pi}{4G_6} Q_5 \quad (4.19)$$

To obtain the H -integral, we can in principle split the interior product of the Killing vector with the three-form as

$$i_K G = d\Lambda + H. \quad (4.20)$$

However, for our purposes we do not need to do this explicitly: the integral of $i_K G$ and of H are identical, as the contribution of $d\Lambda$ for Λ a well-defined one-form cancels anyway.

To make contact with the supersymmetric limit later, we consider the Killing vector

$$K = \partial_t + \alpha \partial_y. \quad (4.21)$$

with α a constant. Then we find that locally

$$d\omega \equiv i_K G|_{t=\text{const.}}, \quad \omega = \frac{c_1}{s_1} H_1^{-1} \left(dy + \left(\frac{c_p}{s_p} - \alpha \right) k + B^{(p)} \right) - \frac{c_1}{s_1} dy + \left(\frac{s_p}{c_p} - \alpha \right) B^{(1)}. \quad (4.22)$$

The one-form ω is zero at infinity and well-behaved at any finite distance, but note that it is not globally well-defined. The integral $\int_{\mathbb{R}^2} i_K G_3$ only receives a contribution from the origin $r = r_+$. A short computation shows that for constant $\tilde{\psi}, \tilde{\phi}$:

$$B^{(i)}|_{r=r_+} = -\frac{s_p c_p}{s_i c_i} dy, \quad k|_{r=r_+} = 0. \quad (4.23)$$

and hence the first bracket in (4.22) does not contribute in the \mathbb{R}^2 -integral. The other terms give:

$$\int_{\mathbb{R}^2} H = \int_{\mathbb{R}^2} i_K G = -L_y \omega_y|_{r=r_+} = L_y \left(\frac{c_1}{s_1} + \frac{s_p^2 - \alpha s_p c_p}{s_1 c_1} \right) = L_y \frac{M_1 + M_p - \alpha Q_p}{Q_1}, \quad (4.24)$$

using the notation

$$M_i = \frac{m}{2} \cosh(2\delta_i), \quad (4.25)$$

which gives the contribution to the 5D ADM mass in the i -channel (so that $M_{ADM,5D} = (L_y \pi)/(4G_6) \sum_i M_i$).

In the end, we find that (4.17) becomes

$$\mathcal{Q}_K = -\frac{L_y \pi}{4G_6} \left(\frac{M_5 + M_p - \alpha Q_p}{Q_5} Q_5 + \frac{M_1 + M_p - \alpha Q_p}{Q_1} Q_1 \right) \quad (4.26)$$

$$= -\frac{L_y \pi}{4G_6} (M_1 + M_5 + 2(M_p - \alpha Q_p)). \quad (4.27)$$

For $\alpha = 0$, we have $K = \partial_t$ and we retrieve the 6D ADM mass (4.3) for the Komar charge \mathcal{Q}_K . Note that each term of the second line contributes to the M_p -channel. Also, in a sense, the non-extremality resides only in the integral over H ; the integrals over S^3 of G_3, \tilde{G}_3 contribute the charge. For $\alpha = 1$, so that $K = \partial_t + \partial_y$, the Komar charge in the supersymmetric limit becomes the usual null charge $\mathcal{Q}_K = -(L_y \pi)/(4G_6)(Q_1 + Q_5)$.

5 Discussion and Outlook

Fluxes on non-trivial topology can support stationary configurations. This is a feature much used in microstate geometries and explained in detail in [13] for five-dimensional smooth microstates. We have explored the six-dimensional guise of this mechanism for horizonless solutions. The three-form field strengths of six-dimensional supergravity and the 2-, 3-cohomology play a crucial role and give a non-trivial contribution to the Komar integral (2.25) and thus to the conserved charges.

Many other avenues remain unexplored. One interesting direction is to explore the Smarr formula and the role of topology for non-flat asymptotics. As we have seen, compact directions give brane-like interpretations to the Komar integrals in terms of energy and tension densities. It would be interesting to understand the extension to asymptotic Anti-de Sitter spaces. In string theory, spaces of the asymptotic form $AdS_p \times S^q$ are very common. For these geometries, one must take care to regulate the Komar integral and perform a suitable background subtraction for the infinite AdS background contribution and render the Komar integrals finite. However, it does not seem that this subtraction term would be expressible in terms of an interesting topological integral. In six dimensions, supersymmetric microstate solutions have $AdS_3 \times S^3$ core regions, and one can reinterpret our results for these geometries in their own right. In fact, for the D1-D5 solutions of section 3.3 it is clear that the relevant (non-trivial) three-cycle will be the S^3 and the two-cycle will be the ($t = \text{const.}$) non-compact spatial two-cycle of AdS_3 . In other words, besides the subtlety of background subtraction, the situation for these AdS_3 geometries will be entirely analogous to the solutions considered here. Perhaps more enlightening would be $AdS_5 \times S^5$ asymptotics, the arena of smooth LLM geometries [47]. The topological contribution to the Smarr formula for 1/16 BPS solutions might also shed light on possible smooth geometries with the asymptotics of the Gutowski-Reall black hole [48, 49].

Perhaps a similar discussion of topology can give us insight into the cosmological horizon. A Smarr formula has been discussed in the past [50, 51], but there has not been a discussion within supergravity models, nor with focus on topology. We leave such investigations, for instance for the de Sitter-Schwarzschild black hole, to future work.

One of the original motivations of this work was to understand how to discriminate between supersymmetric and non-supersymmetric smooth solution with non-zero Hawking temperature. The best studied example of the latter are the JMaRT solutions, which are smooth in six dimensions and hence fit in our current study.¹⁰ They have an ergoregion, which gives rise to an instability [57] that has been connected to Hawking radiation [58, 59]. One might expect that the appearance of an ergoregion in non-extremal microstate geometries is crucial for their decay and the connection to non-extremal black holes. Then one might also expect that the ergoregion plays a role in the universal characterization of microstate geometries through the Komar integral, as topology-supported solitons. However, the ergosurface is not topological and hence does not play a special role in the Komar

¹⁰It would be interesting to study the various known five-dimensional non-extremal constructions, such as those based on JMaRT [52–54] and Bolt-like [55, 56] solutions.

integral. Hence the appearance of an ergoregion in the gravitational back-reaction of the probe constructions [11, 12] remains an open question.

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A Uplift of Five-Dimensional Multi-Center Solutions

A.1 General reduction

Reducing 6D minimal supergravity plus a tensor multiplet gives the STU model in 5D. The 6D metric \hat{g}_{ab} decomposes into the 5D metric g_{ab} , a graviphoton A_a^1 , and a scalar ϕ_2 . The three-form gives two gauge fields: $\hat{G}_{abc} \sim (\star_5 F^2)_{abc}$ and $\hat{G}_{ab6} \sim F_{ab}^3$. Finally, our 6D scalar gives a scalar in 5D $\hat{X} = \phi_1$. We can then reparametrize the 5D scalars ϕ_1, ϕ_2 to get the usual three constrained scalars X^I of the STU model.

We use hats to denote 6D quantities in this section; unhatted quantities, such as indices, are 5D. We start with the 6D Lagrangian:

$$\sqrt{-\hat{g}} \mathcal{L}_6 = \sqrt{-\hat{g}} \left[\hat{R} - 2\partial_{\hat{\mu}} X \partial^{\hat{\mu}} X - \frac{1}{3} e^{2\sqrt{2}\hat{X}} \hat{G}_{\hat{\mu}\hat{\nu}\hat{\rho}} \hat{G}^{\hat{\mu}\hat{\nu}\hat{\rho}} \right]. \quad (\text{A.1})$$

We call the (spacelike) coordinate along which we reduce y . The reduction ansatz for the metric is:

$$d\hat{s}^2 = e^{\phi_2/\sqrt{6}} ds_5^2 + e^{-3\phi_2/\sqrt{6}} (dy + A_a^1 dx^a)^2, \quad (\text{A.2})$$

with inverse:

$$(\partial\hat{s})^2 = e^{-\phi_2/\sqrt{6}} (\partial s_5)^2 - 2e^{-\phi_2/\sqrt{6}} A^{1\mu} \partial_\mu \partial_y + (e^{3\phi_2/\sqrt{6}} + e^{-\phi_2/\sqrt{6}} (A^1)^2) \partial_y^2. \quad (\text{A.3})$$

The Einstein-Hilbert Lagrangian then reduces to:

$$\frac{1}{G_6} \sqrt{-\hat{g}} \hat{R} = \frac{1}{G_5} \sqrt{-g} \left[R - \frac{1}{2} (\partial\phi_2)^2 - \frac{1}{4} e^{-4\phi_2/\sqrt{6}} (F^1)^2 \right], \quad (\text{A.4})$$

where $G_6 = L_y G_5$. Note that $\sqrt{-\hat{g}} = e^{\phi_2/\sqrt{6}} \sqrt{-g_5}$.

The kinetic term for the 6D scalar \hat{X} gives the contribution:

$$\frac{1}{G_6} \sqrt{-\hat{g}} \left[-2\partial_{\hat{\mu}} X \partial^{\hat{\mu}} X \right] = \frac{1}{G_5} \sqrt{-g} \left[-2(\partial\phi_1)^2 \right]. \quad (\text{A.5})$$

Finally, reducing the three-form can be done most easily using form notation. The reduction ansatz is:

$$2\hat{G} = e^{-2\sqrt{2}\phi_1+2\phi_2/\sqrt{6}} \star_5 F^3 + F^2 \wedge (dy + A^1), \quad (\text{A.6})$$

which also implies:

$$2\hat{\star}\hat{G} = e^{2\phi_2/\sqrt{6}} \star_5 F^2 + e^{-2\sqrt{2}\phi_1} F^3 \wedge (dy + A^1). \quad (\text{A.7})$$

Then the reduction of the kinetic term is:

$$2e^{2\sqrt{2}X}\hat{\star}\hat{G} \wedge \hat{G} = dy \wedge \left[\frac{1}{2}e^{-2\sqrt{2}\phi_1+2\phi_2/\sqrt{6}} F_3 \wedge \star_5 F_3 + \frac{1}{2}e^{2\sqrt{2}\phi_1+2\phi_2/\sqrt{6}} \star_5 F_2 \wedge F_2 + F^3 \wedge F^2 \wedge A^1 \right]. \quad (\text{A.8})$$

Summarizing, the reduction gives us the 5D Lagrangian:

$$\begin{aligned} \sqrt{-g} \mathcal{L}_5 = \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi_2)^2 - \frac{1}{4}e^{-4\phi_2/\sqrt{6}\phi_2}(F^1)^2 - 2(\partial\phi_1)^2 \right. \\ \left. - \frac{1}{4}e^{2\sqrt{2}\phi_1+2\phi_2/\sqrt{6}}(F^2)^2 - \frac{1}{4}e^{-2\sqrt{2}\phi_1+2\phi_2/\sqrt{6}}(F^3)^2 \right] - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma\lambda} A_\mu^1 F_{\nu\rho}^2 F_{\sigma\lambda}^3. \end{aligned} \quad (\text{A.9})$$

To bring this to the usual STU form, we can define:

$$X_1 = e^{2\phi_2/\sqrt{6}}, \quad X_2 = e^{-\phi_2/\sqrt{6}-\sqrt{2}\phi_1}, \quad X_3 = e^{-\phi_2/\sqrt{6}+\sqrt{2}\phi_1}, \quad (\text{A.10})$$

so that $X^1 X^2 X^3 = 1$, and the Lagrangian can be written as:

$$\mathcal{L}_5 = R - \frac{1}{4} \frac{1}{(X^I)^2} (F^I)^2 - \frac{1}{2} \frac{(\partial X^I)^2}{(X^I)^2} - \frac{1}{4} e^{-1} \epsilon^{\mu\nu\rho\sigma\lambda} A_\mu^1 F_{\nu\rho}^2 F_{\sigma\lambda}^3, \quad (\text{A.11})$$

with sum over $I = \{1, 2, 3\}$ implied. This is the usual form of the STU Lagrangian. We can also write this as:

$$\mathcal{L}_5 = R - \frac{1}{2} Q_{IJ} F_{\mu\nu}^I F^{J\mu\nu} - Q_{IJ} \partial_\mu X^I \partial^\mu X^J - \frac{1}{24} e^{-1} C_{IJK} \epsilon^{\mu\nu\rho\sigma\lambda} A_\mu^I F_{\nu\rho}^J F_{\sigma\lambda}^K, \quad (\text{A.12})$$

where we have $C_{IJK} = |\epsilon_{IJK}|$ and:

$$\frac{1}{6} C_{IJK} X^I X^J X^K = 1, \quad (\text{A.13})$$

$$Q_{IJ} := \frac{9}{2} X_I X_J - \frac{1}{2} C_{IJK} X^K, \quad (\text{A.14})$$

$$X_I := \frac{1}{6} C_{IJK} X^J X^K. \quad (\text{A.15})$$

A.2 Uplifting SUSY solutions

The most general 6D supersymmetric metric can be written as [35, 40]:

$$ds_6^2 = -2H^{-1}(dv + \beta)[du + \omega + \frac{\mathcal{F}}{2}(dv + \beta)] + Hdx_4^2, \quad (\text{A.16})$$

$$= -H^{-1}\mathcal{F}[dv + \beta + \mathcal{F}^{-1}(du + \omega)]^2 + H^{-1}\mathcal{F}^{-1}(du + \omega)^2 + Hdx_4^2. \quad (\text{A.17})$$

The rewriting of the metric in the second line shows us that we can reduce along v as long as it is a spacelike coordinate, i.e. $\mathcal{F} < 0$ everywhere. The reduction gives us:

$$\begin{aligned} ds_5^2 &= -H^{-4/3}\mathcal{F}^{-2/3}(du + \omega)^2 + H^{2/3}(-\mathcal{F}^{1/3})dx_4^2, \\ e^{-3\phi_2/\sqrt{6}} &= H^{-1}(-\mathcal{F}), \\ A^1 &= \beta + \mathcal{F}^{-1}(du + \omega). \end{aligned} \quad (\text{A.18})$$

We see that the 6D null coordinate u becomes a timelike coordinate in 5D [35].

With the metric, gauge fields and scalars in 5D given by (3.5)-(3.7), we can then identify the appropriate 6D quantities in terms of the 5D ones as follows:

$$\mathcal{F} = -Z_1, \quad \omega = k, \quad \beta = B^1, \quad H = (Z_2 Z_3)^{1/2}. \quad (\text{A.19})$$

For reference, the full 6D fields are given by:

$$\begin{aligned} ds_6^2 &= -\frac{1}{Z_1(Z_2 Z_3)^{1/2}}(du + k)^2 + (Z_2 Z_3)^{1/2}ds_4^2 + \frac{Z_1}{(Z_2 Z_3)^{1/2}}(dv - Z_1^{-1}(du + k) + B^1)^2, \\ e^{\sqrt{2}X} &= e^{\sqrt{2}\phi_1} = X_1^{1/2}X_3 = \frac{Z_2^{1/2}}{Z_3^{1/2}}, \\ 2G &= X_3^{-2} \star_5 F^3 + F^2 \wedge (dv + A^1). \end{aligned} \quad (\text{A.20})$$

B Rigid T^4 Reduction of IIB and $SO(1, 2)$ Truncation

The reduction of IIB supergravity to six-dimensional $\mathcal{N} = (1, 0)$ supergravity with 2 tensor multiplets goes in two steps. In a first step, reduction of the bosonic sector on a rigid T^4 gives a theory with $SO(2, 2)$ global symmetry [60]. Then the compatibility with D1-D5-P supersymmetries as in [22] leads to the bosonic sector of the $SO(1, 2)$ invariant supergravity.

First, we reduce IIB supergravity on a T^4 , keeping only the components of the fields with indices over the remaining six dimensions. This gives us two dilatons (from the 10D dilaton ϕ and the breathing mode of the T^4); two axions (from the 10D axion $C_{(0)}$ and from the only relevant component of $C_{(4)}$), along with the two reduced three-forms coming from the potentials $C_{(2)}$ and $B_{(2)}$. The reduction ansatz is [60, 61]:

$$\begin{aligned} ds_{10, str}^2 &= e^{\phi_1/2} \left(e^{\phi_2/2} ds_6^2 + e^{-\phi_2/2} ds_{T_4}^2 \right), & C_{(0)} &= \chi_1, \\ \phi &= \phi_1, & C_{(2)} &= C_{(2)}, \\ B_{(2)} &= B_{(2)}, & C_{(4)} &= -\chi_2 \text{vol}(T_4) + \dots, \end{aligned} \quad (\text{B.1})$$

where $ds_{T_4}^2$ and $\text{vol}(T^4)$ are the flat metric and flat volume element on T^4 . The \dots in $C_{(4)}$ are other terms that follow from the self-duality condition $F_{(5)} = \star F_{(5)}$. Note that we use the IIB supergravity conventions as in [31]. The resulting 6D Lagrangian is [60]:

$$\begin{aligned} \mathcal{L}_{6D, SO(2,2)} &= R - \frac{1}{2}(\partial\phi_1)^2 - \frac{1}{2}(\partial\phi_2)^2 - \frac{1}{2}e^{2\phi_1}(\partial\chi_1)^2 - \frac{1}{2}e^{2\phi_2}(\partial\chi_2)^2 \\ &\quad - \frac{1}{12}e^{-\phi_1-\phi_2}H_{(3)}^2 - e^{\phi_1-\phi_2}\frac{1}{12}F_{(3)}^2 + \chi_2 H_{(3)} \wedge dC_{(2)}, \end{aligned} \quad (\text{B.2})$$

with $F_{(3)} \equiv dC_{(2)} - C_{(0)}H_{(3)}$. This reduction/truncation has an $SO(2, 2) \cong SL(2)_1 \times SL(2)_2$ symmetry where each $\tau_i = \chi_i + ie^{-\phi_i}$ parametrizes an $SL(2)/SO(2)$ coset. The $SO(2, 2)$ is not a symmetry of the tensor Lagrangian, but rather of the equations of motion and Bianchi identities. Those can be written as Bianchi identities of an $SO(2, 2)$ vector of field strengths G^r with components

$$\begin{pmatrix} G^1 \\ G^2 \end{pmatrix} \equiv \begin{pmatrix} dB_{(2)} \\ dC_{(2)} \end{pmatrix}, \quad \begin{pmatrix} G^3 \\ G^4 \end{pmatrix} \equiv \begin{pmatrix} \frac{d\mathcal{L}_{6D}}{dG^2} \\ -\frac{d\mathcal{L}_{6D}}{dG^1} \end{pmatrix} = -e^{\phi_2}(i\sigma_2) \cdot \mathcal{M}_1 \cdot \begin{pmatrix} \star G^1 \\ \star G^2 \end{pmatrix} + \chi_2 \begin{pmatrix} G^1 \\ G^2 \end{pmatrix}. \quad (\text{B.3})$$

Those tensors obey the duality relation (compare (2.19)):

$$\mathcal{M}_{rs}G^s = \eta_{rs} \star G^s, \quad (\text{B.4})$$

with the off-diagonal $SO(2, 2)$ metric $\eta = (i\sigma_2) \otimes (i\sigma_2)$ and scalar matrix

$$\mathcal{M} = \mathcal{M}_2(\tau_2) \otimes \mathcal{M}_1(\tau_1), \quad \text{with} \quad \mathcal{M}_i = V_i V_i^T, \quad V_i = \begin{pmatrix} e^{-\frac{1}{2}\phi_i} & \chi_i e^{\frac{1}{2}\phi_i} \\ 0 & e^{\frac{1}{2}\phi_i} \end{pmatrix}. \quad (\text{B.5})$$

It is important to realize that this $SO(2, 2)$ theory cannot be the bosonic part of any supergravity theory. One can perform a further truncation to obtain a theory that can be the bosonic part of $SO(1, 2) \cong SL(2)$ supergravity by setting $\tau_2 = f(\tau_1)$ with f an $SL(2)$ -transformation. This identifies a ‘diagonal’ $SL(2)$ subgroup in $SO(2, 2) \cong SL(2)_1 \times SL(2)_2$. The four tensors G^r then decompose in a singlet and a triplet under this truncation. Consistency of the truncation requires that we put the singlet to zero.

We are interested in solutions with the supersymmetries of the D1-D5-P system [22], giving the truncation:

$$\tau_2 = -\frac{1}{\tau_1}. \quad (\text{B.6})$$

The τ_2 equation of motion then requires that we put the singlet $G^1 + G^4$ to zero. The remaining three field strengths are

$$\hat{G}^1 = \frac{1}{2}(G^3 - G^2), \quad \hat{G}^2 = \frac{1}{2}(G^2 + G^3), \quad \hat{G}^3 = \frac{1}{2}(G^4 - G^1), \quad (\text{B.7})$$

Dropping the hats again, G^r then obeys the self-duality relation with the $SO(1, 2)$ matrix

$$V = \exp(\chi E_+) \exp(\phi H/2), \quad E_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (\text{B.8})$$

and the Komar integral (2.25) applies.

To make the connection to the theory with one tensor multiplet clear, we write the vanishing singlet as an anti self-duality constraint on a three-form G' :

$$G' = -\star G', \quad G' = dB_{(2)} + \chi_2 dC_{(2)} = \frac{e^{-2\phi_1}H_{(3)} - \chi_1 F_{(3)}}{e^{-2\phi_1} + \chi_1^2}. \quad (\text{B.9})$$

We can then take $G \equiv (1/2)dC_{(2)}$ to be the (unrestricted) tensor that is the combination of the other self-dual and anti self-dual tensors. In section 3.4, we take $\phi = \phi_1, \chi = \chi_1$ and $B = (1/2)C_{(2)}, B' = B_{(2)}$. An obvious further truncation of this $SO(1, 2)$ theory is to take $G' = 0, \chi_1 = 0$ which leaves us with the $SO(1, 1)$ sector used in large parts of this paper, after the identification $\phi_1 = \sqrt{2}X$.

For the $SO(1, 2)$ theory with the unrestricted three-form G and the anti self-dual three-form G' as defined above, the generalization (2.24) of (2.29) for the null charge reduces to:

$$\mathcal{Q}_K = -\frac{1}{4\pi G_6} \int_V \left(H \wedge \tilde{G} + \tilde{H} \wedge G \right) + \frac{1}{8\pi G_6} \int_V (H' \wedge G'), \quad (\text{B.10})$$

where H, \tilde{H} are defined as in (2.30), keeping in mind the $SO(1, 2)$ -generalized definitions for the dual form:

$$\tilde{G} = \frac{e^{2\phi_1}}{1 + e^{2\phi_1}\chi_1^2} \star G, \quad (\text{B.11})$$

The harmonic form H' is defined by the split:

$$i_K G' = d\Lambda' + H', \quad (\text{B.12})$$

where Λ' is a globally defined one-form.

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